B U L L E T I N

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ON FIBONOMIAL AND OTHER TRIANGLES VERSUS DUALITY TRIADS

Summary

   The duality triads were defined in the preceding paper. Notation, enumeration of formulas and references is therefore to be continued hereby. In this paper Fibonomial triangles [3–8] and further Pascal-like triangles including the q-Gaussian one are given explicit interpretation as discrete time dynamical systems as it is in the case with duality triads. Large classes of duality triads in various areas of mathematics are identified including those triads that appear in finite operator calculus and its extensions [9–17].

1.   New examples of dual triads and related general information

The duality triads were defined in the preceding paper [22].

1.1. q-Gaussian dual triad

(see: pp. 68-74, Proposition 12.1, recurrence 12.4, Propositions 12.3 and 12.4 in [32]). Let us recall that q-Gaussian [32, 33, 34,16] coefficient is given by

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n_q!}{k_q!(n-k)_q!} = \left(\begin{array}{c} n \\ n-k \end{array} \right)_q.$$

For a positive integer $q > 1; q = p^m$ where p is prime $\left(\begin{array}{c} n \\ k \end{array} \right)_q$ is interpreted [32, 35] as the number of k-dimensional subspaces in n-th dimensional space over Galois field $GF(q)$. Also q real and $-1 < q < +1$ are exploited in vast literature (see: (2.5)

in [33] and references therein). Here for

$$n > 0, \quad n_q \equiv \frac{1 - q^n}{1 - q}$$

and while using the following notation for q-factorial (see also [34, 17]):

$$n_q! = n_q(n-1)_q!; \quad 1_q! = 0_q! = 1, \quad n_q^{\underline{k}} = n_q(n-1)_q \ldots (n-k+1)_q,$$

we write

$$\left(\begin{array}{c} n \\ k \end{array} \right)_q \equiv \frac{n_q^{\underline{k}}}{k_q!}.$$

The recurrence relation [16, 32, 33] for connection constants

$$C_{n,k} = \left(\begin{array}{c} n \\ k \end{array} \right)_q$$

reads as follows:

$$\left(\begin{array}{c} n+1 \\ k \end{array} \right)_q = q^k \left(\begin{array}{c} n \\ k \end{array} \right)_q + \left(\begin{array}{c} n \\ k-1 \end{array} \right)_q,$$

(1)

$$\left(\begin{array}{c} n \\ 0 \end{array} \right)_q = 1, \quad n \geq 0, \quad k \geq 1$$
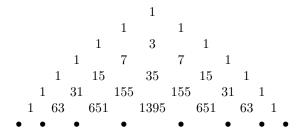
and might be considered as the result of the q-Leibniz rule in q-umbral calculus (p. 176 in [16]) of Eulerian formal power series [12, 32, 35]. Another equivalent form is the following

$$\left(\begin{array}{c} n+1 \\ k \end{array} \right)_q = \left(\begin{array}{c} n \\ k \end{array} \right)_q + q^{n-k} \left(\begin{array}{c} n \\ k-1 \end{array} \right)_q,$$
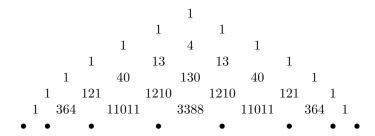
$$\left(\begin{array}{c} n \\ 0 \end{array} \right)_q = 1, \quad n \geq 0, \quad k \geq 1.$$

(The Eulerian formal power series as well as many others form algebras isomorphic to the reduced incidence algebras [12, 32, 35]). The above recurrences (1) may be represented by the q-Gaussian triangles. The corresponding $q = 2$, $q = 3$, $q = 5$ cases are supplied below:

(I) $q = 2$

```
                            1
                        1       1
                    1       3       1
                1       7       7       1
            1      15      35      15      1
        1      31     155     155      31      1
    1     63     651    1395     651     63     1
    •    •      •        •        •      •     •    •
```

(II) $q = 3$

```
                                    1
                           1                 1
                  1                 4                 1
         1                 13                13                 1
1                 40                130                40                 1
         1        121       1210              1210       121       1
1        364       11011           3388           11011       364       1
   •     •        •              •             •              •       •    •
```

(III) $q = 5$

```
                                    1
                           1                 1
                  1                 6                 1
         1                 31                31                 1
1                 156               806               156                 1
         1        781       20306             20306       781       1
1        3906      508431           16401          508431      3906       1
   •     •        •              •             •              •       •    •
```
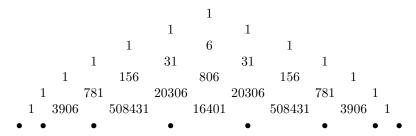
The recurrence dual to (1) is then given by

$$n\Phi_n(x) = q^n \Phi_n(x) + \Phi_{n+1}(x),$$

(2)

$$\Phi_0(x) = 1, \quad \Phi_{-1}(x) = 0, \quad ; \quad n \geq 0.$$

in accordance with a well known fact [32] that

(3)
$$x^n = \sum_{k \geq 0} \binom{n}{k}_q \Phi_k(x),$$

$$\Phi_k(x) = \prod_{s=0}^{k-1} (x - q^s).$$

$\Phi_k(x) = \prod_{s=0}^{k-1} (x - q^s)$ are named to be q-Gaussian polynomials and note that these are monic persistent root polynomials [26] therefore the recurrence (1) is of the form

$$[r] = \left\{ \left(q^{k-1} \right) \right\}_{k \geq 1},$$

$$L_{n+1,k} = L_{n,k-1} + r_{k+1} L_{n,k},$$

(4)

$$L_{0,0} = 1, \quad L_{0,-1} = 0, \quad n, k \geq 0,$$

in the notation of [26] i.e.

$$\left(\begin{array}{c} n \\ k \end{array} \right)_q = c_{n,k} = L_{n,k}.$$

1.2. Catalan triad and triangle

Let $C_{n,k}$ denotes the number of pairs of nonintersecting paths of length n and distance k as defined in [36]. Then one may prove [36] that $C_{n,k}$ (where now $n \geq k > 0$) satisfy the recurrence

$$C_{n+1,k} = C_{n,k-1} + 2C_{n,k} + C_{n,k+1},$$

(5)

$$C_{1,1} = 1, \quad C_{n,0} = 0 = C_{n,n+k} = 0 \quad \text{for} \quad n \geq k \geq 0.$$

The transition matrix is then of the form

$$X = \left(\begin{array}{ccccccc} 2 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 2 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 2 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 2 & 1 & \ldots \end{array} \right)$$

and we obtain the Catalan triangle calculating its subsequent rows $C_n = C_0 X^n; n \geq 0$ in accordance with the fact [36] that

$$C_{n,k} = \left(\begin{array}{c} 2n \\ n-k \end{array} \right) \frac{k}{n} :$$

(note:

$$C_{n,1} = \left(\begin{array}{c} 2n \\ n-1 \end{array} \right) \frac{1}{n} = \left(\begin{array}{c} 2n \\ n \end{array} \right) \frac{1}{n+1}$$

are Catalan numbers).

The Catalan Triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & & 1 & & & & & \\
 & & & & 0 & & 2 & & 1 & & & & \\
 & & & 0 & & 5 & & 4 & & 1 & & & \\
 & & 0 & & 14 & & 14 & & 6 & & 1 & & \\
 & 0 & & 42 & & 48 & & 27 & & 8 & & 1 & \\
 \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet
\end{array}
$$

with dual recurrence for the corresponding polynomials being of the form

$$xC_k(x) = C_{k+1}(x) + 2C_k(x) + C_{k-1}(x), \quad k \geq 0$$

(6)

$$C_0(x) = 1, \quad C_{-1}(x) = 0.$$

To complete the triad we note according to general rule that

$$(7) \qquad x^n = \sum_{1 \le k \le n} \binom{2n}{n-k} \frac{k}{n} C_k(x), \quad n > 0.$$

We call the polynomial sequence $\{C_k(x)\}_{k \ge 0}$ the Catalan polynomial sequence.

1.3. Remarks and Information IV

The q-Gaussian polynomials

$$\Phi_k(x) = \prod_{s=0}^{k-1} (x - q^s)$$

from Example 7 (along with some other examples) constitute a persistent root polynomial sequence and therefore as such they do satisfy the recurrence equation [23] (see: Remark and Information I)

$$L_{n+1,k} = L_{n,k-1} + r_{k+1} L_{n,k},$$

(8)

$$L_{0,0} = 1, \quad L_{0,-1} = 0 \quad n, k \ge 0,$$

where $[r] \equiv \{r_0, r_1, r_2, \ldots, r_k, \ldots\}$ is the root sequence determining $\{\Phi_n\}_{n \ge 0}$ and the connection constants $(L_{n,k})$ – called in [23] the generalized Lah numbers – are determined by duality triad (7) equation

$$(9) \qquad x^n = \sum_{0 \le k \le n} L_{n,k} \Phi_k(x), \quad n \ge 0,$$

thus completing another example of duality triad. Naturally this is the case with any persistent root polynomial sequence $\{\Phi_n\}_{n \ge 0}$. On this important occasion note that because of Farward Theorem (see: p. 21 in [29] and [26]) persistent root polynomial sequences do not form OPS sequences (OPS = Orthonormal Polynomial System – [29]) because OPS sequences should satisfy three term recurrences. An example of OPS at hand is provided by Catalan polynomials satisfying (6). The other examples are classical Hermite or Laguerre polynomials or Tchebychev OPS (in this last case – X represents the symmetric Jacobi matrix see: p. 138 in [29]).

Concluding: persistent root polynomials sequences $\{\Phi_n\}_{n \ge 0}$ and (disjointedly) OPS sequences – orthogonal with respect to quasi-definite moment functional – give rise to corresponding duality triads. Some of OPS sequences like Hermite, Laguerre, q-Hermite [16, 37], q-Laguerre (see: (33) in [38] and [17]) etc. are generalized Appell polynomials.

Note 1: not all Appell polynomials give rise to duality triads (see: Remarks and Information I).

Note 2: all generalized Appell polynomials that are at the same time persistent root polynomials are discovered by the authors of [26] and these polynomial sequences – naturally – give rise to duality triads.

2.   Does Fibonomial dual triad exist?

Let us start investigating this problem with reference to q-Gaussian dual triad example. The corresponding recurrence relation connection constants $c_{n,k}$ might be considered as the result of the q-Leibniz rule in q-umbral calculus (p. 176 in [16]) of Eulerian formal power series [12, 32, 35], where that series as well as many others form algebras isomorphic to the standard reduced incidence algebras [12, 32, 35].
The connection constants $c_{n,k} = \begin{pmatrix} n \\ k \end{pmatrix}_q$ from relation (9) of the triad under consideration are interpreted [12, 32, 35] as the number of k-dimensional subspaces in n-th dimensional space $V(n,q)$ over the Galois field $GF(q)$. Restating this in incidence algebras language [35] we conclude that in the lattice $L(n,q)$ of all subspaces of $V(n,q)$:

$n_q!$ = *the number of maximal chains in an of interval of length* n,

$$\begin{pmatrix} n \\ k \end{pmatrix}_q = \begin{bmatrix} \alpha \\ \beta\gamma \end{bmatrix} :=$$

the number of distinct elements z *in a segment* $[x, y]$ *of type* α *such that*
$[x, z]$ *is of type* β *while* $[z, y]$ *is of type* γ,

where $x, y, z \in L(n,q)$ and the type of $[x, y]$ is equal to $\dim[x/y]$. (Here $[y/x]$ denotes the subtraction of linear subspaces [35]).

In straightforward analogy consider now *Fibonomial coefficients* [3–8], [39–41] $c_{n,k}$, given by

$$c_{n,k} = \begin{pmatrix} n \\ k \end{pmatrix}_F = \frac{F_n!}{F_k! F_{n-k}!} = \begin{pmatrix} n \\ n-k \end{pmatrix}_E,$$

where [34, 43]:

$$n_F \equiv F_n \neq 0, \quad n_F! \equiv n_F(n-1)_F(n-2)_F(n-3)_F \ldots 2_F 1_F; \quad 0_F! = 1;$$

$$n_F^{\underline{K}} = n_F(n-1)_F \ldots (n-k+1)_F; \quad \begin{pmatrix} n \\ k \end{pmatrix}_F \equiv \frac{n_F^{\underline{k}}}{k_F!}.$$

A natural question then arises:

Does there exist a partialy ordered set P *with its incidence algebra* $R(P)$ *and such incidence coefficients* $\begin{bmatrix} \alpha \\ \beta\gamma \end{bmatrix}$ *that coincide with* F-*binomial ones i.e.* $\begin{bmatrix} \alpha \\ \beta\gamma \end{bmatrix} = \begin{pmatrix} n \\ k \end{pmatrix}_F$?

For the moment this is an open problem for us. However independently of that one may naturally consider – in analogy to Eulerian formal series algebra – also a

"Fibonomial" formal series algebra with a Fibonomial convolution as the associative product [43, 26, 48], i.e.

$$c_n = \sum_{k \geq 0}^{n} \binom{n}{k}_F a_k b_{n-k}, \quad \text{where} \quad F(z)G(z) = H(z) \equiv \sum_{n \geq 0} \frac{c_n}{n!} z^n,$$

$$G(z) \equiv \sum_{n \geq 0} \frac{b_n}{n!} z^n \quad \text{and} \quad F(z) \equiv \sum_{n \geq 0} \frac{a_n}{n!} z^n.$$

The corresponding Fibonomial finite operator calculus [43] becomes then an example of the general theory of the "calculus of sequences" [34] in its operator form [13–15], [49]. As for the recurrence (see: p. 27,formule (58) in [41]) for *Fibonomial coefficients* $c_{n,k}$, it is well known [39–41] ([42] is also relevant) and some properties of Fibonomial numbers $\binom{n}{k}_F$ i.e. of Fibonomial triangle were among others considered in [3–8], [40], [44–48]. This recurrence reads:

$$\binom{n+1}{k}_F = F_{k-1} \binom{n}{k}_F + F_{n-k+2} \binom{n}{k-1}_F,$$

(10)

$$\binom{n}{0}_F = 1, \quad \binom{0}{k}_F = 0 \quad \text{for} \quad k > 0$$

or, equivalently,

$$\binom{n+1}{k}_F = F_{k+1} \binom{n}{k}_F + F_{n-k} \binom{n}{k-1}_F,$$

$$\binom{n}{0}_F = 1, \quad \binom{0}{k}_F = 0 \quad \text{for} \quad k > 0.$$

Note that the coefficients of the recurrence (10) for $c_{n,k}$ depend on both k and n as it was not the case in all previous examples. We face now a new situation out of the scope of the former definition of dual triad, where it was assumed that in

$$c_{n+1,k} = i_{k-1} c_{n,k-1} + q_k c_{n,k} + d_{k+1} c_{n,k+1}$$

(11)

$$c_{0,0} = 1; \quad c_{0,k} = 0 \quad \text{for} \quad k > 0.$$

the numbers i_k, q_k, d_k; $k \geq 0$ – are *independent* of "discrete time" parameter n.

This assumption was decisive in deriving the third completing member of the triad. Nevertheless – it seems – we still may maintain a "discrete time" dependent interpretation. Namely $c_{n,k} = $ *number of ways to reach level k in n steps starting from the level* 0 and still (10) may be described as such a discrete time system in which corresponding rows ("states") of the $C = (c_{n,k})$ matrix (hence: rows of Fibonomial triangle) are reached via consecutive application of different, appropriately

adjusted, unique one step transition matrix F, which however is not that of dual recurrence – simply, because dual recurrence does not exist. Nevertheless, generalizing [1] we still may maintained the standpoint the kl-th entry of $F(n) = $ *number of ways of going from the level $k \to$ to $\to l$ in n steps* where $(c_{n,k} \neq 0$ for $k \leq n)$ F is the solution of recurrent for rows of C equation $CF = EC$ from which it follows immediately that $E^n C = CF^n$, $n \geq 0$, where

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} = (\delta_{i+1,k})_{i,k \geq 0}.$$

Such a matrix F exists – in our case – giving rise in a standard way to the Fibonomial triangle

$$
\begin{array}{ccccccccc}
 & & & & \mathbf{1} & & & & \\
 & & & \mathbf{1} & & \mathbf{1} & & & \\
 & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \\
 & \mathbf{1} & & 2 & & 2 & & \mathbf{1} & \\
 \mathbf{1} & & \mathbf{3} & & \mathbf{6} & & 3 & & \mathbf{1} \\
\end{array}
$$

$$
\begin{array}{ccccccccccc}
\mathbf{1} & & 5 & & \mathbf{15} & & 15 & & 5 & & \mathbf{1} \\
\mathbf{1} & & 8 & & 40 & & 60 & & 40 & & 8 & & \mathbf{1} \\
\bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet \\
\end{array}
$$

which, as said, might be obtained when considering subsequent powers of this triangle generating matrix F

$$X = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 2 & 1 & 1 & 0 & \dots \\ 0 & -2 & 0 & 6 & 2 & 1 & \dots \end{pmatrix}$$

and then calculating $C_n = C_0 F^n$; $n \geq 0$, where $C_0 = (1,0,0,0,\dots)$. The subsequent rows of F are:

$$0,\ 2,\ -10,\ 0,\ 15,\ 3,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\dots$$
$$0,\ 36,\ 16,\ -80,\ -100,\ 40,\ 5,\ 1,\ 0,\ 0,\ 0,\ 0,\dots$$
$$0,\ \dots,\ \dots,\ \dots,\ \dots,\ \dots,\ \dots,\ 8,\ 1,\ 0,\ 0,\ 0,\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

One here just adjusts F matrix elements step by step while calculating $C_n = C_0 F^n$ The formula for matrix elements of F is beyond our abilities and needs. The formula for the k-th component $[C_n]_k$ of C_n is tautologically obvious:

$$[C_n]_k = \binom{n}{k}_F.$$

Eventually expected recurrence in the form of an eigenvector and eigenvalue equation (6) $x\Phi = F\Phi$ by no means would be dual recurrence to the recurrence (10). The resulting sequence Φ is not a polynomial sequence (deg $p_n = n$). We may see it immediately from the start:

$$\Phi_0(x) = 1, \quad \Phi_1(x) = x - 1, \quad \Phi_2(x) = 0, \quad \Phi_3(x) = 1 - x,$$
$$\Phi_4(x) = -(x-1)^2, \quad \Phi_5(x) = (2-x)(x-1)^2 + 8(x-1).$$

Moreover any polynomial sequence solution would be dual recurrence:

$$x\Phi_k(x) = d_k(n)\Phi_{k-1}(x) + q_k(n)\Phi_k(x) + i_k(n)\Phi_{k+1}(x);$$

(12)

$$\Phi_0(x) = 1, \quad \Phi_{-1}(x) = 0; \quad k \geq 0,$$

which does not exist for numbers i_k, q_k, d_k; $k \geq 0$ – *dependent* of "discrete time" parameter n as it is the case with the Fibonomial coefficients recurrence.

Conclusion. Neither dual recurrences nor triad polynomials exist in the sense of numbers for i_k, q_k, d_k; $k \geq 0$ being *dependent* of "discrete time" parameter n. The Fibonomial case under consideration is one of such examples as

(13) $$c_{n,k} = \left[\begin{array}{c} n \\ k \end{array}\right], \quad c_{n,k} = \left\langle\begin{array}{c} n \\ k \end{array}\right\rangle, \quad c_{n,k} = \left(\begin{array}{c} n \\ k \end{array}\right)_F.$$

3. Families of triad polynomials

Existence of dual triad is the quite specific property of corresponding objects as there are many polynomial sequences of distinguished importance which are triad polynomial sequences as well as many polynomial sequences of distinguished importance which are not triad polynomial sequences: for example Abel, Euler or exponential polynomials.

Here we summarize the information supplied above as to *when* a family of polynomial sequences $\{\Phi_n\}_{n\geq 0}(\deg\Phi_n(x) = n)$ is a family of triad polynomials.

1. First of all let us recall that (11) and (12) imply (13) but not vice versa as (13) does not imply (11) (Remarks and Information I).

2. Secondly for numbers i_k, q_k, d_k; $k \geq 0$ – *dependent* of "discrete time" parameter n in the recurrence (11) triads do not exist. Famous examples are:

$$c_{n,k} = \left[\begin{array}{c} n \\ k \end{array}\right], \quad c_{n,k} = \left\langle\begin{array}{c} n \\ k \end{array}\right\rangle, \quad c_{n,k} = \left(\begin{array}{c} n \\ k \end{array}\right)_F.$$

The polynomial sequence solution would be dual recurrence (12) which does not exists for numbers i_k, q_k, d_k; $k \geq 0$ – dependent of "discrete time" parameter n.

3. Any persistent roots polynomial sequence is a sequence of triad polynomials (All generalized Appell polynomials that are at the same time persistent root polynomials are given in [26]).

4. Any OPS – with respect to quasi-definite moment functional – is a sequence of triad polynomials.

5. All OPS being at the same time generalized Appell polynomial sequences have been determined by Chihara in [50] (see also [29] p. 167 and [51] and [52], [9, 12, 29], [53]).

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PORÓWNANIE TRÓJKĄTÓW FIBONOMIALNYCH I POKREWNYCH Z TRÓJKAMI DUALNYMI

S t r e s z c z e n i e

W pracy poprzedzającej wprowadzono pojęcie triad dualnych, o których pełna informacja może być przedstawiona w postaci "trójkąta" na podobieństwo trójkąta Pascala. W niniejszej pracy będącej kontynuacją poprzedniej rozważa się trójkąt fibonomialny [3–8], q-Gaussowski i inne interpretowane jako układy dynamiczne z czasem dyskretnym. W pracy zidentyfikowano dwie obszerne klasy triad reprezentowane poprzez ciągi wielomianów quasi-ortogonalnych (w tym ortogonalnych) [29] oraz te reprezentowane przez ciągi wielomianów znanych pod nazwą "persistent roots polynomials" [26]. Pośród tych przykładów znajdujemy interesujące przypadki z klasycznego i rozszerzonego skończonego rachunku operatorowego Roty [12].